WHERE IS THE WATER TABLE? A REASSESSMENT OF THE DUPUIT-FORCHHEIMER THEORY USING HIGHER ORDER CLOSURE HYPOTHESIS

O. Castro-Orgaz[1], J.V. Giráldez[1,2] and L. Mateos[1]

[1]Instituto de Agricultura Sostenible. CSIC. Dept. of Agronomy, Alameda del Obispo, 14004 Cordoba, España. e-mail: oscarcastro@ias.csic.es
[2]University of Cordoba. Dept. of Agronomy, Cra Madrid km 396, 14071 Cordoba, España.

RESUMEN. Según las hipótesis de DF, el agua subterránea fluye en el interior de la zona saturada con líneas de corriente paralelas siempre que la pendiente de la superficie freática sea pequeña. Esto conduce a un modelo unidimensional sencillo con el que se determina la posición de la superficie freática, que es el contorno superior de la red de corriente. Sin embargo, estas hipótesis han sido aceptadas sin un análisis riguroso. En este trabajo se presentan las ecuaciones de Dupuit-Fawer, que permiten caracterizar la posición de la superficie freática cuando las líneas de corriente no son rectilíneas. Dichas ecuaciones se obtienen mediante una aproximación en series de la Laplaciana de la función de corriente, usando coordenadas curvilíneas. El modelo propuesto se ilustra mediante simulaciones numéricas para el caso de flujo de drenaje por zanjas abiertas. La comparación muestra la superioridad de las ecuaciones de Dupuit-Fawer sobre las de Dupuit-Forchheimer, sin que la mayor precisión exija un incremento de complejidad apreciable.

ABSTRACT. Watertable depth has been usually estimated using a simplified groundwater flow equation based on the Dupuit-Forchheimer (DF) hypotheses. According to DF hypotheses, groundwater streamlines are parallel as far as the watertable slope is small. These hypotheses lead to a simple one-dimensional flow equation valid to determine watertable depth. Nevertheless the validity of these hypotheses have not been well explored. The purpose of this report is the presentation of the Dupuit-Fawer equations obtained from the Laplace equation expressed in curvilinear coordinates, after a power series expansion of some of its terms. This model indicates that DF equations are a particular case of the general formulation with restrictions beyond small local slope and horizontal streamlines. The proposed model is applied to the case of open surface drainage ditch. The comparison showed that Dupuit-Fawer equations are more accurate than the DF model despite similar complexity.

1.- Introduction

The solution of numerical models for the unsaturated zone requires the determination of the water table position, that is the boundary where the humidity reaches 100%. A theoretical model for the computation of the water table position is described herein following Castro-Orgaz (2011).

Steady two-dimensional flow in saturated porous media (Fig. 1) obeys Darcy's law (Bear 1972)

$$u = -K\frac{\partial}{\partial s}\left[\frac{p}{\gamma}+z\right] = -K\frac{\partial \phi}{\partial s} \qquad (1)$$

where u is flow velocity in streamline direction s, K is the saturated hydraulic conductivity, p/γ is pressure head, $z = z_b+y$ is the elevation, z_b the bed elevation, y the vertical coordinate above it, s is the curvilinear coordinate in streamline direction and $\phi = p/\gamma+z$ is potential function.

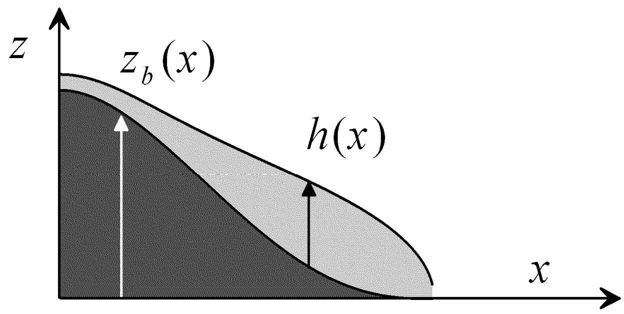

Fig. 1. Definition of two-dimensional phreatic flow in porous media

The continuity equation in the Cartesian x and z directions can be expressed by Laplace's equation in terms of ϕ as (Bear 1972)

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial z^2} = 0 \qquad (2)$$

The Cauchy-Riemann equations state

$$\frac{\partial \psi}{\partial z} = K\frac{\partial \phi}{\partial x}, \quad \frac{\partial \psi}{\partial x} = -K\frac{\partial \phi}{\partial z} \qquad (3)$$

where ψ is the stream function, that also satisfies Laplace's equation

$$\nabla^2\psi = \frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial z^2} = 0 \qquad (4)$$

Two-dimensional flow computations may be avoided if streamlines are nearly parallel. Then, an approximation similar to the classical gradually-varied flow theory in open channels is valid (Jaeger 1956)

$$u = -K\frac{\mathrm{d}h}{\mathrm{d}x} \tag{5}$$

where h is flow depth and u velocity in the x-direction. This approximation was first proposed by Jules Dupuit in 1863 (Fig. 2) and it is still today widely accepted in groundwater hydraulics (Hager 2004).

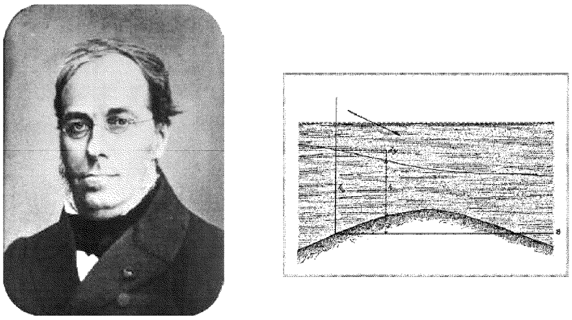

Fig. 2. Jules Dupuit and a sketch of his theory of 1863

Implicit in Eq. (5) is that u is constant with depth and it has no vertical component. Equation (5) also follows from (1) with $\phi = h$ at the free surface and $s \to x$. However, as for gradually-varied surface flow, Eq. (5) fails if the curvature of seepage streamlines is notable. Then the full two-dimensional potential flow approach is required (Muskat 1942, Polubarinova-Kochina 1962).

In this work, an alternative model is presented. The model is based on the theory developed by Fawer (Castro-Orgaz 2011) to describe the variation of streamline curvature from the channel bottom to the free surface in a vertical section of steady free surface flow. This approximation, originally proposed by Boussinesq for the momentum principle, reduces the two-dimensional Euler equations to a pseudo-one-dimensional hydraulic approach. Applied to groundwater flow through porous media, this approximation is an alternative to the Dupuit-Forchheimer model that avoids the full solution of Laplace's equation. The new model is applied to seepage flow in earth dams. The current theory can be extended to other groundwater flow problems considering surface accretion, anisotropy or axe-symmetric flow (Bear 1972, Knight 2005).

2.- Approximate treatment of two-dimensional groundwater flow

Consider steady flow in porous media over an arbitrarily curved and sloped impermeable layer $z_b = z_b(x)$ (Fig. 1). The Dupuit-Fawer equations are the potential flow equation derived from Euler equations in natural, curvilinear (s,n) coordinates (Rouse 1938)

$$\frac{\partial u}{\partial n} = \kappa u \tag{6}$$

where κ is streamline curvature and n distance in a trajectory normal to s. The local radius of curvature R is related to the inclination of the plane curve with respect to the horizontal α by

$$\frac{1}{R} = \kappa = -\frac{\partial \alpha}{\partial s} \tag{7}$$

with α clockwise positive. Extending Eq (3), the Cauchy-Riemann equations in these coordinates are

$$u = -\frac{\partial \psi}{\partial n} = -K\frac{\partial \phi}{\partial s}, \quad \frac{\partial \psi}{\partial s} = -K\frac{\partial \phi}{\partial n} = 0 \tag{8}$$

The integration of Eq. (6) along an equipotential curve yields (Rouse 1938)

$$u = u_s \exp\left(-\int_n^N \frac{\mathrm{d}n}{R}\right) \tag{9}$$

with u_s as free surface (subscript s) velocity and N length of an equipotential curve. The relation between the curvature radius and streamline inclination and their corresponding values at the boundary bottom and free surface may be assumed. For example (Hager and Hutter 1984, Castro-Orgaz 2011)

$$\frac{1}{R} = \frac{1}{R_b} + \left(\frac{1}{R_s} - \frac{1}{R_b}\right)\nu^m \tag{10}$$

$$\alpha = \alpha_b + (\alpha_s - \alpha_b)\nu \tag{11}$$

with $\nu = n/N$ the dimensionless curvilinear coordinate along a normal, m an exponent and α the streamline inclination (Eq. 7). Equations (10) and (11) are the base for reducing the two-dimensional problem to one-dimensional problem. The unknown free surface position is replaced by an explicit expression involving the curvature of the boundary streamline (Hager and Hutter 1984). The case m = 1 was suggested by Hager and Hutter (1984). Using Eq. (10), Eq. (9) is integrated as (Hager and Hutter 1984)

$$u = u_s \exp\left\{\frac{N}{R_s}\left[\chi(\nu - 1) + (1 - \chi)\frac{(\nu^{m+1} - 1)}{m+1}\right]\right\} \tag{12}$$

where $\chi = R_s/R_b$ is relative curvature. The discharge rate Q through an equipotential is, using Eq. (12),

$$Q = \int_0^N \frac{\partial \psi}{\partial n} \mathrm{d}n = \int_0^N u \mathrm{d}n \approx N u_s \left[1 - \frac{N}{R_s} \left(\frac{\chi}{2} + \frac{1-\chi}{m+2} \right) \right] \quad (13)$$

where the velocity distribution given by Eq. (12) is approximated by a truncated power series expansion of the exponential function to first order (Hager and Hutter 1984). Further, the relative free surface curvature is

$$\frac{1}{R_s} = \left[\frac{d^2}{dx^2} \left(h + z_b \right) \right] \left\{ 1 + \left[\frac{d}{dx} \left(h + z_b \right) \right]^2 \right\}^{-3/2} \quad (14)$$

$$\frac{h}{R_s} = \frac{hh'' + hz_b''}{\left(1 + \left(h' + z_b' \right)^2 \right)^{3/2}} \quad (15)$$

where primes denote derivatives with respect to x. At the bottom streamline, Eq. (15) becomes

$$\frac{h}{R_b} = \frac{hz_b''}{\left(1 + z_b'^2 \right)^{3/2}} \quad (16)$$

N and h may be related by assuming that the shape of the equipotential is a circular arc, from which, for a Taylor power series expansion in small α, follows (Hager and Hutter 1984)

$$\frac{h}{N} = 1 - \frac{3z_b'^2 + 3z_b'h' + 2h'^2}{6} \quad (17)$$

Inserting Eqs. (15) to (17) into Eq. (13) and retaining only first order terms, it results in an expression for the free surface velocity (Castro-Orgaz 2011). Boussinesq-type equations are mathematically valid for weak streamline curvature and slope, e.g. $|hh''|$, $|hz''|$, z'^2, $|z'h'|$ and $h'^2 < 0.5$ (Hager and Hutter 1984). The latter relates to the two-dimensional potential flow analogy in seepage flow using Fawer's theory for steady flows with curved streamlines. Using Eq. (1), an additional condition for the free surface velocity is obtained

$$u_s = -K \frac{\partial}{\partial s} \left[\frac{p}{\gamma} + z \right]_s = -K \frac{\partial}{\partial s} \left(h + z_b \right) \quad (18)$$

Further, $\sin\alpha_s$ (Fig. 1) may be expressed as a function of the free surface streamline inclination $\mathrm{d}(h+z_b)/\mathrm{d}x$ using trigonometric relations

$$\sin \alpha_s = \left(h' + z_b' \right) \cos \alpha_s = \frac{h' + z_b'}{\left[1 + \left(h' + z_b' \right)^2 \right]^{1/2}} \quad (19)$$

which together with Eq. (17) produces

$$u_s = -K \sin \alpha_s = -K \frac{h' + z_b'}{\left[1 + \left(h' + z_b' \right)^2 \right]^{1/2}} \quad (20)$$

a relation that included the head loss of the porous media after incorporating Darcy's law in its formulation. The potential flow analogy in porous media permits to equal Eq. (20) and the free surface velocity from the curved flow, resulting

$$\frac{Q}{Kh} \left[1 + \frac{hh''}{(m+2)\left(1 + h'^2 \right)^{3/2}} - \frac{h'^2}{6} \right] + \frac{h'}{\left(1 + h'^2 \right)^{1/2}} = 0 \quad (21)$$

which is a second-order non-linear differential equation describing two-dimensional flow in porous media over an arbitrary, impervious bottom geometry $z_b = z_b(x)$. This equation can be solved numerically provided two boundary conditions are specified. If a shooting method, *e.g.* the Runge-Kutta scheme, is selected, then values of h and h' at a given section are required. If a two-point boundary value scheme is adopted, then values of h at the two boundary sections are the input data. Equation (21) is called the *Dupuit-Fawer* equation in honour of Fawer.

3.- Application to the drainage ditch

A practical case to evaluate Eq. (21) is the two-dimensional flow in earth dams over a horizontal, impervious bottom, for which the bottom geometry is $z_b = z_b(x) = 0$. Considering the simplest situation $h/R_s \approx hh''$ and $m = 1$, it results in

$$\frac{Q}{Kh} \left(1 + \frac{hh''}{3} - \frac{h'^2}{6} \right) \left(1 + h'^2 \right)^{1/2} + h' = 0 \quad (22)$$

Expanding the square-root term in Eq. (22) by a Taylor power series and retaining first order terms results in

$$\frac{Q}{Kh} \left(1 + \frac{hh'' + h'^2}{3} \right) + h' = 0 \quad (23)$$

If streamlines were parallel, Eq. (23) reduces to the Dupuit expression (Jaeger 1956)

$$\frac{Q}{Kh} + h' = 0 \quad (24)$$

This is the classical problem treated by Polubarinova-Kochina (1962), discussed by Jaeger (1956) as flow into a ditch (Fig. 3a). Extensive studies by Chapman (1957) and Polubarinova-Kochina (1962) concluded that discharge rate Q is accurately given by the Dupuit-Forchheimer equation

$$Q = \frac{K}{2L}\left(h_e^2 - h_w^2\right) \tag{25}$$

where h_e is upstream flow depth, h_w is downstream flow depth and L is length of dam. Herein, Eq. (25) is used in Eq. (23). The flow in a rectangular earth dam for h_e/L = 1 and h_w/h_1= 0 (Knight 2005) is considered in Fig. 3b. Equation (23) was solved numerically using the standard 4th-order Runge-Kutta method with the boundary conditions $h/h_e(x/h_e$= 0)=1 and $h'(x/h_e$= 0)=0. The results are compared in Fig. 3b with the numerical solution of Laplace equation (2), showing good agreement.

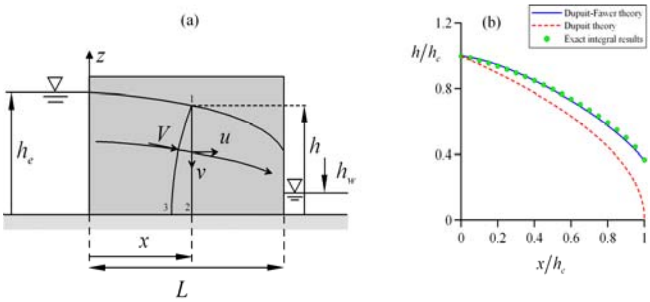

Fig. 3. Flow in rectangular drainage ditch. (a) Definition sketch. (b) Test case for h_e/L=1 and h_w/h_e = 0

In contrast, the standard Dupuit parabola is

$$\frac{h^2}{2} = \frac{h_e^2}{2} - \frac{Q}{K}x \tag{26}$$

The Dupuit solution is worse than the two-dimensional solution as Fig. 3b, shows The Dupuit-Fawer solution is very close to the numerical solution.

4.- Discussion of results

Although the implications of Eq.(23) with respect to the Dupuit approximation are not too evident in the differential equation form, some analytical developments will explain better the differences between them . Equation (23) can be written in the form

$$1 + \frac{1}{3}\frac{\mathrm{d}}{\mathrm{d}x}\left(hh'\right) = -hh'\frac{K}{Q} \tag{27}$$

After a double integration subjected to the boundary conditions $h/h_e(x/h_e$= 0)=1 and $h'(x/h_e$= 0)=0, the the seepage surface $h = h(x)$ is

$$h = \left[-\frac{2}{3}\left(\frac{Q}{K}\right)^2 \exp\left[-\left(\frac{1}{3}\frac{Q}{K}\right)^{-1} x\right] + 2\frac{Q}{K}\left(\frac{1}{3}\frac{Q}{K} - x\right) + h_e^2\right]^{1/2} \tag{28}$$

It can be observed that the existence of streamline curvature and slope effects is accounted for in the general analytical solution by the exponential functions. These are neglected, however, in the standard parabolic Eq.(26). The improved formulation proposed with the Dupuit-Fawer equations can be applied to some of the hydrogeological problems where the Dupuit hypotheses fail, like in the vicinity of a well.

5.-Conclusions

Fawer's theory for steady, curved-streamline, potential flow in open channels has been successfully extended to curved-streamline, potential flow of groundwater seepage through isotropic and homogeneous porous media. The new equation generalizes the equation of Dupuit. The potential flow analogy was used to equate this result with those established by using Darcy's law to account for head loss in porous media, resulting in a non-linear second-order differential equation describing the seepage surface over arbitrary impervious boundaries. The resulting equation, called *Dupuit-Fawer* equation, is demonstrated to be a generalized form of the classical Dupuit-Forchheimer equation that allows for streamline curvature effects. The *Dupuit-Fawer* equation incorporates two-dimensional flow features into a one-dimensional model, skipping two-dimensional potential flow equations.

As a practical application of the *Dupuit-Fawer* equations, seepage flow across earth dams was considered. A simplified model for weakly-curved and -sloped streamlines was developed for discharge calculation and applied to the rectangular dam problem, comparing successfully with full two-dimensional potential flow methods.

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